

一类二阶半线性微分方程的振动性

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摘要:运用 Riccati 变换, 针对 $\alpha > \beta$ 和 $\beta \geq \alpha$ 两种情况研究了一类二阶半线性微分方程的振动性, 得到该方程振动的充分条件。

关键词:微分方程; 振动准则; Riccati 变换

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Oscillation of a class of second order half-linear differential equation

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Abstract: In this paper, the oscillation of a class of second order half-linear differential equation for both $\alpha > \beta$ and $\beta \geq \alpha$ are studied by using the Riccati transformation. The sufficient condition for the oscillation of the equation is obtained.

Key words: differential equation; oscillation criteria; Riccati transformation

1 引言

微分方程有着悠久的历史。随着科学技术的发展, 微分方程理论在自然科学和社会科学领域均具有广泛的应用。对于微分方程的振动性而言, 若其解是振荡的, 则方程的解就是振动的。近年来, 微分方程振动性理论作为微分方程定性理论的重要内容之一, 受到人们的普遍关注, 其发展相当迅速, 从线性到非线性, 从低阶到高阶, 从分散到连续, 都有一些新的研究成果。因此, 对其进行研究不仅具有重要的理论意义, 也具有重要的实际应用价值。文献[1]研究了二阶微分方程

$$(r(t)(z'(t))^{\alpha})' + q(t)f(x(\sigma(t))) = 0$$

的振动性, 其中 $\frac{f(u)}{u^{\alpha}} \geq k$ 。

文献[2]研究了 n 阶微分方程

$$\frac{d}{dt}[r(t)\Phi(z^{(n-1)}(t))] + \int_a^b c(t, \xi)\Phi(x(g(t, \xi)))d\sigma(\xi) = 0$$

的振动性, 其中, n 为偶数, $\Phi(s) = |s|^{p-2}s$, $p > 1$ 。

文献[3]研究了二阶微分方程

$$(r(t)|u'(t)|^{\alpha-1}u'(t))' + p(t)|u(\tau(t))|^{\alpha-1}u(\tau(t)) = 0.$$

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的振动性。

文献[4-6]研究了二阶微分方程的振动性,但文献[1-6]都只考虑了微分方程中两项的次数相等(即 $\alpha = \beta$)的情形,是按照常规的 Riccati 变换方法得到微分方程振动的充分条件。文献[7-16]也研究 $\alpha = \beta$ 的情况,将所作的二阶方程 Riccati 变换推广到相应的三阶微分方程,得到了三阶微分方程振动的充分条件。但由于文献[1-6]研究二阶方程时是对 $\alpha = \beta$ 情形进行了研究,因此,在将二阶微分方程的 Riccati 变换推广到三阶微分方程时,只能针对三阶微分方程 $\alpha = \beta$ 的情形进行了研究^[7-16]。可是,振动性问题不能仅仅只研究 $\alpha = \beta$ 的特殊情形,对于 $\alpha > \beta$ 和 $\beta \geq \alpha$ 的情形,如果还采用常规的 Riccati 变换方法就不行了。文献[17]在改进了文献[1-16]所作的常规 Riccati 变换技巧后,得到了方程

$$(r(t) | z'(t) |^{a-1} z'(t))' + q(t) | x(\sigma(t)) |^{\beta-1} x(\sigma(t)) = 0$$

在 $\alpha \geq \beta > 0$ 情形振动的充分条件。文献[18]对文献[17]中采用的 Riccati 变换和积分平均技巧作进一步的推广,对文献[17]研究的方程在两种情况下得到了该方程振动的充分条件,推广和改进了文献[17]的方法和结论。

在文献[18]的基础上,研究如下的带积分项的二阶半线性中立型微分方程:

$$(r(t) | z'(t) |^{a-1} z'(t))' + \int_c^d q(t, \xi) x^\beta(\sigma(t, \xi)) d\xi = 0, \quad t \geq t_0. \quad (1)$$

为了能对 $\alpha > \beta$ 和 $\beta \geq \alpha$ 两种情况都能进行研究,我们推广文献[18]的 Riccati 变换和积分平均技巧,利用推广的广义 Riccati 变换和积分平均技巧,得到方程(1)振动的充分条件。

对方程(1), $z(t) = x(t) + \int_a^b p(t, \xi) x(\tau(t, \xi)) d\xi$, $0 \leq a < b$, $0 \leq c < d$, α 和 β 是正常数。根据文献[1, 3, 6, 9],还需满足下列条件:

H₁) $r(t) \in C^1([t_0, \infty), \mathbf{R}_+)$, $r'(t) \geq 0$, $r(t) > 0$;

H₂) $\tau(t, \xi) \in C([t_0, \infty) \times [a, b], \mathbf{R}_+)$, $\tau(t, \xi) \leq t$, $\lim_{t \rightarrow \infty} \tau(t, \xi) = \infty$;

H₃) $\sigma(t, \xi) \in C^1([t_0, \infty) \times [c, d], \mathbf{R}_+)$, $\sigma(t, \xi) \leq t$, $\sigma_1(t) = \sigma(t, d)$, $\sigma'_1(t) > 0$, $\lim_{t \rightarrow \infty} \sigma(t, \xi) = \infty$,
 $\sigma(t, \xi)$ 关于变量 ξ 是减函数;

H₄) $p(t, \xi) \in C([t_0, \infty) \times [a, b], \mathbf{R}_+)$, $q(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbf{R}_+)$, $p(t) = \int_a^b p(t, \xi) d\xi$ 。

2 主要结论

针对方程(1)中出现的 α 和 β 的情形,对 $\alpha > \beta$ 和 $\beta \geq \alpha$ 两种情况研究方程(1)的振动,在文献[18]的基础上,应用推广的 Riccati 变换和积分平均技巧,得到方程(1)在 $\alpha > \beta$ 和 $\beta \geq \alpha$ 两种情况下振动的充分条件。

定理 1 设

$$\int_{t_0}^{\infty} r^{-\frac{1}{a}}(t) dt = \infty, \quad (2)$$

且存在 $\rho(t) \in C^1([t_0, \infty), \mathbf{R}_+)$,使得

$$\int_{t_0}^{\infty} [\rho(t) Q(t) - \frac{\lambda^\lambda r(t) (\rho'(t))^{\lambda+1}}{(\lambda+1)^{\lambda+1} k^\lambda (\rho(t) \sigma'_1(t))^\lambda}] dt = \infty, \quad (3)$$

其中, $\lambda = \min\{\alpha, \beta\}$, $Q(t) = \int_c^d q(t, \xi) [1 - p(\sigma(t, \xi))]^\beta d\xi$, $\sigma_1(t) = \sigma(t, d)$, 常数 $k > 0$, 则方程(1)振动。

证明: 用反证法。假设方程(1)有非振动解 $x(t)$,不妨设 $x(t) > 0$,由方程(1),有

$$(r(t) | z'(t) |^{a-1} z'(t))' = - \int_c^d q(t, \xi) x^\beta(\sigma(t, \xi)) d\xi < 0, \text{ 即}$$

$$(r(t) | z'(t) |^{a-1} z'(t))' < 0.$$

则 $r(t) | z'(t) |^{a-1} z'(t)$ 是减函数, $z'(t)$ 定号,故 $z'(t) > 0$ 或 $z'(t) < 0$ 。

断言 $z'(t) > 0$ 。否则 $z'(t) < 0$,有

$$(r(t) | z'(t) |^{a-1} z'(t))' = (-r(t) (-z'(t))^a)' < 0. \text{ 则}$$

$$(r(t)(-z'(t))^a)' > 0.$$

即 $r(t)(-z'(t))^a$ 是增函数, 则存在 $T > 0$, 对 $\forall t > T$, 有

$$r(t)(-z'(t))^a \geq r(T)(-z'(T))^a = k_1,$$

其中 $k_1 = r(T)(-z'(T))^a$, 则

$$z'(t) \leq -\left(\frac{k_1}{r(t)}\right)^{\frac{1}{a}}.$$

将上式在 $[T, t]$ 上积分, 得

$$0 < z(t) \leq Z(T) - k_1^{\frac{1}{a}} \int_T^t r^{-\frac{1}{a}}(s) ds.$$

上式与式(2)矛盾, 因此, $z'(t) > 0$.

由 $z(t)$ 的定义, $z(t) \geq x(t)$, 则 $z(\tau(t, \xi)) \geq x(\tau(t, \xi))$, 而 $z'(t) > 0$, $z(t)$ 是增函数, $z(\tau(t, \xi)) \leq z(t)$, 故

$$z(t) \leq x(t) + \int_a^b p(t, \xi) z(\tau(t, \xi)) d\xi \leq x(t) + z(t) \int_a^b p(t, \xi) d\xi \leq x(t) + z(t)p(t).$$

即

$$x(t) \geq z(t)[1 - p(t)], x(\sigma(t, \xi)) \geq z(\sigma(t, \xi))[1 - p(\sigma(t, \xi))].$$

而 $\sigma(t, \xi)$ 关于变量 ξ 是减函数, 有 $\sigma(t, \xi) \geq \sigma(t, d)$, 故 $z(\sigma(t, \xi)) \geq z(\sigma(t, d))$, 则

$$x(\sigma(t, \xi)) \geq z(\sigma(t, d))[1 - p(\sigma(t, \xi))].$$

则

$$\int_c^d q(t, \xi) x^\beta(\sigma(t, \xi)) d\xi \geq z^\beta(\sigma(t, d)) \int_c^d q(t, \xi) [1 - p(\sigma(t, \xi))]^\beta d\xi.$$

由式(1), 得

$$(r(t)(z'(t))^a)' + z^\beta(\sigma(t, d)) \int_c^d q(t, \xi) [1 - p(\sigma(t, \xi))]^\beta d\xi \leq 0.$$

令 $\sigma_1(t) = \sigma(t, d)$, $Q(t) = \int_c^d q(t, \xi) [1 - p(\sigma(t, \xi))]^\beta d\xi$, 则

$$(r(t)(z'(t))^a)' + Q(t)z^\beta(\sigma_1(t)) \leq 0. \quad (4)$$

定义函数

$$w(t) = \rho(t) \frac{r(t)(z'(t))^a}{z^\beta(\sigma_1(t))}.$$

则

$$\begin{aligned} w'(t) &= \rho'(t) \frac{r(t)(z'(t))^a}{z^\beta(\sigma_1(t))} + \rho(t) \frac{(r(t)(z'(t))^a)'}{z^\beta(\sigma_1(t))} - \rho(t) \frac{\beta r(t)(z'(t))^a z'(\sigma_1(t)) \sigma_1'(t)}{z^{\beta+1}(\sigma_1(t))} \\ &\leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) Q(t) - \rho(t) \frac{\beta r(t)(z'(t))^a z'(\sigma_1(t)) \sigma_1'(t)}{z^{\beta+1}(\sigma_1(t))}. \end{aligned}$$

1) 若 $\beta \geq \alpha$, 则

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) Q(t) - \frac{\beta \sigma_1'(t) (z(\sigma_1(t)))^{\frac{\beta-\alpha}{a}} z'(\sigma_1(t))}{(\rho(t)r(t))^{\frac{1}{a}}} w^{\frac{a+1}{a}}(t).$$

由于 $z'(t) > 0$, $z(t)$ 是增函数, 而 $\sigma_1'(t) > 0$, 则对 $\forall t > T$, 有 $(z(\sigma_1(t)))^{\frac{\beta-\alpha}{a}} \geq (z(\sigma_1(T)))^{\frac{\beta-\alpha}{a}} = k_2$. 而 $(r(t)(z'(t))^a)' \leq 0$, 即 $r(t)(z'(t))^a$ 是减函数, 有 $r(t)(z'(t))^a \leq r(\sigma_1(t))(z'(\sigma_1(t)))^a$. 即

$$\frac{z'(\sigma_1(t))}{z'(t)} \geq \left(\frac{r(t)}{r(\sigma_1(t))}\right)^{\frac{1}{a}}.$$

则

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) Q(t) - \frac{\beta k_2 \sigma_1'(t)}{(\rho(t)r(\sigma_1(t)))^{\frac{1}{a}}} w^{\frac{a+1}{a}}(t).$$

由条件 H_1) 和 H_3), $r(\sigma_1(t)) \leq r(t)$, 因此, 有

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) Q(t) - \frac{\beta k_2 \sigma'_1(t)}{(\rho(t)r(t))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(t). \quad (5)$$

2) 若 $\alpha > \beta$, 则

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) Q(t) - \frac{\beta \sigma'_1(t) z'(\sigma_1(t))}{(\rho(t)r(t))^{\frac{1}{\beta}} (z'(t))^{\frac{\alpha}{\beta}}} w^{\frac{\beta+1}{\beta}}(t).$$

由于 $(r(t)(z'(t))^\alpha)' \leq 0$, 得 $z''(t) \leq 0$, 故 $z'(t)$ 是减函数, 有 $z'(\sigma_1(t)) \geq z'(t)$, 则

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) Q(t) - \frac{\beta \sigma'_1(t)}{(\rho(t)r(t))^{\frac{1}{\beta}} (z'(t))^{\frac{\alpha}{\beta}-1}} w^{\frac{\beta+1}{\beta}}(t).$$

又 $z'(t)$ 是减函数, 则对 $\forall t > T$, 有

$$(z'(t))^{\frac{\alpha}{\beta}-1} \leq (z'(T))^{\frac{\alpha}{\beta}-1} = k_3^{-1}.$$

则

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) Q(t) - \frac{\beta k_3 \sigma'_1(t)}{(\rho(t)r(t))^{\frac{1}{\beta}}} w^{\frac{\beta+1}{\beta}}(t). \quad (6)$$

综上所述, 令 $\lambda = \min\{\alpha, \beta\}$, $k = \min\{\beta k_2, \beta k_3\}$, 则式(5)与式(6)合并写成

$$w'(t) \leq -\rho(t) Q(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{k \sigma'_1(t)}{(\rho(t)r(t))^{\frac{1}{\lambda}}} w^{\frac{\lambda+1}{\lambda}}(t).$$

利用不等式 $Bu - Au^{\frac{\lambda+1}{\lambda}} \leq \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \frac{B^{\lambda+1}}{A^\lambda}$, 有

$$w'(t) \leq -\rho(t) Q(t) + \frac{\lambda^\lambda r(t) (\rho'(t))^{\lambda+1}}{(\lambda+1)^{\lambda+1} k^\lambda (\rho(t) \sigma'_1(t))^\lambda}.$$

在 $[T, \infty)$ 上积分, 得

$$0 < w(t) \leq w(T) - \int_T^\infty \left[\rho(s) Q(s) - \frac{\lambda^\lambda r(s) (\rho'(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1} k^\lambda (\rho(s) \sigma'_1(s))^\lambda} \right] ds.$$

此式与式(3)矛盾, 则方程(1)振动。

推论 1 若存在函数 $\rho(t) \in C^1([t_0, \infty), \mathbf{R}_+)$, 使得

$$\int_{t_0}^\infty \rho(t) Q(t) dt = \infty, \int_{t_0}^\infty \frac{r(t) (\rho'(t))^{\lambda+1}}{(\rho(t) \sigma'_1(t))^\lambda} dt < \infty.$$

其中, $Q(t) = \int_c^d q(t, \xi) [1 - p(\sigma(t, \xi))]^\beta d\xi$, $\sigma_1(t) = \sigma(t, d)$ 。则方程(1)振动。

证明: 易验证满足定理 1 条件。

推论 2 设式(2)成立, 若存在 $\rho(t) \in C^1([t_0, \infty), \mathbf{R}_+)$, 使得

$$\int_{t_0}^\infty \frac{r(t) (\rho'(t))^{\lambda+1}}{(\rho(t) \sigma'_1(t))^\lambda} dt = \infty, \quad (7)$$

$$\liminf_{t \rightarrow \infty} \frac{\rho^{\lambda+1}(t) (\sigma'_1(t))^\lambda}{r(t) (\rho'(t))^{\lambda+1}} Q(t) > \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1} k^\lambda} \quad (8)$$

成立, 则方程(1)振动。

证明: 由式(8)知, 存在 $\varepsilon_0 > 0$ 和 $t_1 > t_0$, 使得对 $t > t_1$, 有

$$\frac{\rho^{\lambda+1}(t) (\sigma'_1(t))^\lambda}{r(t) (\rho'(t))^{\lambda+1}} Q(t) > \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1} k^\lambda} + \varepsilon_0.$$

则

$$\rho(t) Q(t) \geq \left(\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1} k^\lambda} + \varepsilon_0 \right) \frac{r(t) (\rho'(t))^{\lambda+1}}{(\rho(t) \sigma'_1(t))^\lambda}.$$

即

$$\rho(t)Q(t) - \frac{\lambda^\lambda r(t)(\rho'(t))^{\lambda+1}}{(\lambda+1)^{\lambda+1} k^\lambda (\rho(t)\sigma'_1(t))^\lambda} \geq \epsilon_0 \frac{r(t)(\rho'(t))^{\lambda+1}}{(\rho(t)\sigma'_1(t))^\lambda}.$$

在 $[t_1, t]$ 上积分, 得

$$\int_{t_1}^t [\rho(s)Q(s) - \frac{\lambda^\lambda r(s)(\rho'(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1} k^\lambda (\rho(s)\sigma'_1(s))^\lambda}] ds > \epsilon_0 \int_{t_1}^t \frac{r(s)(\rho'(s))^{\lambda+1}}{(\rho(s)\sigma'_1(s))^\lambda} ds.$$

由式(7)知满足定理 1 条件, 则方程(1)振动。

定理 2 设式(2)成立, 且

$$\liminf_{t \rightarrow \infty} \frac{1}{Q_1(t)} \int_t^\infty G(s) Q_1^{\frac{\lambda+1}{\lambda}}(s) ds \geq \frac{\lambda}{(\lambda+1)^{\frac{\lambda+1}{\lambda}}}. \quad (9)$$

其中, $G(s) = \frac{\beta c \sigma'_1(s)}{r^{\frac{1}{\lambda}}(\sigma_1(s))}$, $\sigma_1(s) = \sigma(s, d)$, $Q_1(t) = \int_t^\infty Q(s) ds$, $Q(s) = \int_c^d q(s, \xi) [1 - p(\sigma(s, \xi))]^\beta d\xi$, 则方程(1)振动。

证明: 定义函数

$$V(t) = \frac{r(t)(z'(t))^a}{z^\beta(\sigma_1(t))}.$$

则

$$\begin{aligned} V'(t) &= \frac{(r(t)(z'(t))^a)'}{z^\beta(\sigma_1(t))} - \frac{\beta r(t)\sigma'_1(t)(z'(t))^a z'(\sigma_1(t))}{z^{\beta+1}(\sigma_1(t))} \\ &\leq -Q(t) - \frac{\beta r(t)\sigma'_1(t)(z'(t))^a z'(\sigma_1(t))}{z^{\beta+1}(\sigma_1(t))}. \end{aligned}$$

1) 若 $\beta \geq \alpha$, 则

$$V'(t) \leq -Q(t) - \frac{\beta \sigma'_1(t) z'(\sigma_1(t))}{r^{\frac{1}{\alpha}}(t) z'(t)} (z(\sigma_1(t)))^{\frac{\beta-\alpha}{\alpha}} V^{\frac{\alpha+1}{\alpha}}(t).$$

由于 $z'(t) > 0$, $z(t)$ 是增函数, 则对 $\forall t > T$, 有 $(z(\sigma_1(t)))^{\frac{\beta-\alpha}{\alpha}} \geq (z(\sigma_1(T)))^{\frac{\beta-\alpha}{\alpha}} = c_1$. 而 $(r(t)(z'(t))^a)' \leq 0$, 即 $r(t)(z'(t))^a$ 是减函数, 有 $r(t)(z'(t))^a \leq r(\sigma_1(t))(z'(\sigma_1(t)))^a$.

即

$$\frac{z'(\sigma_1(t))}{z'(t)} \geq \left(\frac{r(t)}{r(\sigma_1(t))} \right)^{\frac{1}{a}}.$$

则

$$V'(t) \leq -Q(t) - \frac{\beta c_1 \sigma'_1(t)}{r^{\frac{1}{\alpha}}(\sigma_1(t))} V^{\frac{\alpha+1}{\alpha}}(t). \quad (10)$$

2) 若 $\alpha > \beta$, 则

$$V'(t) \leq -Q(t) - \frac{\beta \sigma'_1(t) z'(\sigma_1(t))}{r^{\frac{1}{\beta}}(t)(z'(t))^{\frac{\alpha}{\beta}}} V^{\frac{\beta+1}{\beta}}(t).$$

而 $(r(t)(z'(t))^a)' \leq 0$, 得到 $z''(t) \leq 0$, $z'(t)$ 是减函数, 有

$$z'(\sigma_1(t)) \geq z'(t).$$

又 $z'(t)$ 是减函数, 则对 $\forall t > T$, 有

$$(z'(t))^{\frac{\alpha}{\beta}-1} \leq (z'(T))^{\frac{\alpha}{\beta}-1} = c_2^{-1}.$$

则

$$V'(t) \leq -Q(t) - \frac{\beta \sigma'_1(t)}{r^{\frac{1}{\beta}}(t)(z'(t))^{\frac{\alpha}{\beta}-1}} V^{\frac{\beta+1}{\beta}}(t) \leq -Q(t) - \frac{\beta c_2 \sigma'_1(t)}{r^{\frac{1}{\beta}}(t)} V^{\frac{\beta+1}{\beta}}(t). \quad (11)$$

综上所述, 令 $\lambda = \min\{\alpha, \beta\}$, $c = \min\{c_1, c_2\}$, 则式(9)与式(10)合并写成

$$V'(t) \leq -Q(t) - \frac{\beta c \sigma'_1(t)}{r^{\frac{1}{\lambda}}(\sigma_1(t))} V^{\frac{\lambda+1}{\lambda}}(t).$$

令 $G(t) = \frac{\beta c \sigma'_1(t)}{r^{\frac{1}{\lambda}}(\sigma_1(t))}$, 则

$$V'(t) + Q(t) + G(t)V^{\frac{\lambda+1}{\lambda}}(t) \leq 0. \quad (12)$$

将式(12)在 $[t, \infty)$ 上积分, 得

$$\int_t^\infty Q(s)ds + \int_t^\infty G(s)V^{\frac{\lambda+1}{\lambda}}(s)ds \leq V(t).$$

令 $Q_1(t) = \int_t^\infty Q(s)ds > 0$, 则上式除以 $Q_1(t)$, 有

$$1 + \frac{1}{Q_1(t)} \int_t^\infty G(s)V^{\frac{\lambda+1}{\lambda}}(s)ds \leq \frac{V(t)}{Q_1(t)}.$$

即

$$\frac{V(t)}{Q_1(t)} \geq 1 + \frac{1}{Q_1(t)} \int_t^\infty G(s)Q_1^{\frac{\lambda+1}{\lambda}}(s) \left(\frac{V(s)}{Q_1(s)} \right)^{\frac{\lambda+1}{\lambda}} ds. \quad (13)$$

由式(9), 存在 $\epsilon_1 > \frac{\lambda}{(\lambda+1)^{\frac{\lambda+1}{\lambda}}}$, 使得

$$\liminf_{t \rightarrow \infty} \frac{1}{Q_1(t)} \int_t^\infty G(s)Q_1^{\frac{\lambda+1}{\lambda}}(s)ds > \epsilon_1 > \frac{\lambda}{(\lambda+1)^{\frac{\lambda+1}{\lambda}}}. \quad (14)$$

令 $\delta = \inf_{t \geq T} \frac{V(t)}{Q_1(t)}$, 则 $\delta \geq 1$, 由式(13), 得

$$\liminf_{t \rightarrow \infty} \frac{V(t)}{Q_1(t)} \geq 1 + \liminf_{t \rightarrow \infty} \frac{1}{Q_1(t)} \int_t^\infty G(s)Q_1^{\frac{\lambda+1}{\lambda}}(s) \left(\frac{V(s)}{Q_1(s)} \right)^{\frac{\lambda+1}{\lambda}} ds.$$

即

$$\delta > 1 + \epsilon_1 \delta^{\frac{\lambda+1}{\lambda}}.$$

则

$$\delta - \epsilon_1 \delta^{\frac{\lambda+1}{\lambda}} > 1. \quad (15)$$

利用不等式 $Bu - Au^{\frac{\lambda+1}{\lambda}} \leq \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \frac{B^{\lambda+1}}{A^\lambda}$, 由(14), 得到

$$\delta - \epsilon_1 \delta^{\frac{\lambda+1}{\lambda}} \leq \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \frac{1}{\epsilon_1^\lambda} < 1.$$

上式与式(15)矛盾, 因此, 方程(1)振动。

注:定理1中的 $\rho(t) \in C^1([t_0, \infty), \mathbf{R}_+)$ 需要根据具体的微分方程去构造, 而定理2没有这个限制条件, 定理2是定理1的补充。定理1是文献[18]中定理2.1的推广, 定理2是文献[18]中定理3.2的推广。

为了说明所得主要结果的适用性, 现通过例题来验证其结论。

例 考虑下列二阶微分方程

$$(t^a |z'(t)|^{a-1} z'(t))' + \int_{3\pi}^{5\pi} \left(\frac{1}{8} x^\beta(t - \frac{\xi}{2}) \right) d\xi = 0. \quad (16)$$

设 $z(t) = x(t) + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{1}{4} x(t - \xi) d\xi$, 这里 $a = \frac{3\pi}{2}$, $b = \frac{5\pi}{2}$, $c = 3\pi$, $d = 5\pi$, $r(t) = t^a$,

$$p(t, \xi) = \frac{1}{4}, \tau(t, \xi) = t - \xi, \sigma_1(t) = \sigma(t, d) = t - \frac{5\pi}{2}, \sigma_1(t) = \sigma(t, d) = t - \frac{5\pi}{2}, q(t, \xi) = \frac{1}{8}.$$

$$\int_{t_0}^\infty r^{-\frac{1}{a}}(t) dt = \int_{t_0}^\infty \frac{1}{t} dt = \infty, Q(t) = \int_{3\pi}^{5\pi} \frac{3^\beta}{2^{2\beta+3}} d\xi = \frac{3^\beta \pi}{2^{2\beta+2}}, \text{ 选取 } \rho(t) = t, \text{ 有}$$

$$\int_{t_0}^\infty \left[\rho(t)Q(t) - \frac{\lambda^\lambda r(t)(\rho'(t))^{\lambda+1}}{(\lambda+1)^{\lambda+1} k^\lambda (\rho(t)\sigma'_1(t))^\lambda} \right] dt = \int_{t_0}^\infty \left[\frac{3^\beta \pi}{2^{2\beta+2}} t - \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1} k^\lambda} \frac{1}{t^\lambda} \right] dt = \infty.$$

故满足定理1条件, 方程(16)振动。

本研究在文献[1-18]对二阶和三阶微分方程振动性研究的基础上,利用推广的广义 Riccati 变换和积分平均技巧,研究了一类二阶半线性中立型微分方程(1),得到了其振动的充分条件,并给出例子说明所得结论的适用性。今后将 Riccati 变换和积分平均技巧应用到更广泛的非线性微分方程模型中。

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