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带有参数和含有 p-Laplacian 算子的混合型分数阶微分系统正解的唯一性

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摘　要:分析了一类带有 p-Laplacian 算子的混合型分数阶微分系统边值问题,其边界条件含有参数,且对参数具有一定的依赖性。利用格林函数的性质和半序 Banach 空间中两个算子之和的不动点定理,推导出正解的唯一性,并构造相应的迭代序列来逼近唯一正解。最后,给出两个数值应用实例验证主要结果的可行性。

关键词:混合型分数阶微分系统;p-Laplacian 算子;和算子;正解;唯一性

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Uniqueness of positive solutions for mixed fractional differential systems with p-Laplacian operators and parameters

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Abstract: This paper analyzes the boundary value problems of a class of mixed fractional differential systems with p-Laplacian operators, whose boundary conditions contain and are dependent on parameters. By using the properties of Green functions and the fixed point theorem of sum of two operators in semi-ordered Banach spaces, the uniqueness of the positive solution is derived and the corresponding iterative sequence to approximate the unique positive solution is constructed. Finally, two numerical examples are given as applications to demonstrate the feasibility of the main results.

Key words: mixed fractional differential system; p-Laplacian operators; sum operator; positive solution; uniqueness

分数阶微分方程在控制、多孔介质、电磁等领域应用广泛,其中含有 p-Laplacian 算子的分数阶微分方程与分数阶微分系统应用于各种自然现象的建模中,可以描述非牛顿流体中非线性现象、建立复杂的过程模型,引起了人们极大地关注。当边界条件不含参数时,已有文献得出了多种边值问题正解的情况[1-11],文献[2]运用混合单调算子的不动点定理研究了带有 p-Laplacian 算子的混合型奇异分数阶微分方程,证明了该问题正解的唯一性;文献[3]运用锥拉伸锥压缩不动点定理得到了带有 p-Laplacian 算子的混合型分数阶微分系统多个正解的存在性;文献[4]利用 Leggett-Williams 不动点定理获得了带有 p-Laplacian 算子和分数阶导数项的耦合分数阶微分系统至少存在 3 个正解的充分条件。当边界条件含有参数时,文献[12]研究了一类带有参数的非线性分数阶多点边值问题非负解的唯一性和存在性;文献[13]研究了边界条件带有参数的分数阶微分方程多点边值问题,运用和算子的性质证明了该问题正解的存在性和唯一性。当运用和算子

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的不动点定理来研究各种分数阶微分方程边值问题正解的存在性与唯一性时，文献[14]运用 γ-凹算子与 γ-凸算子不动点定理证明了分数阶微分系统多点边值问题正解的唯一性并得到迭代序列；文献[15]研究了无穷区间上分数阶微分方程多点边值问题，运用算子之和不动点定理证明了正解的存在唯一性；文献[16]基于和算子的不动点定理证明了带有 p-Laplacian 算子的分数阶微分方程 Riemann-Stieltjes 积分边值问题局部存在的唯一正解，且构造了两个迭代序列来逼近唯一正解。

研究者大多运用非线性泛函分析分数阶微分方程，而运用和算子的不动点定理进行带有参数与 p-Laplacian 算子的分数阶微分系统的研究较少。本研究的创新之处体现在：基于文献[2]，将微分方程变成微分系统，并在边界条件添加参数，且对参数具有依赖性，构造的算子与文献[14-15]有一定的区别；利用和算子的不动点定理研究带有 p-Laplacian 算子的混合型分数阶微分系统。

本研究考虑如下带有参数和含有 p-Laplacian 算子的混合型分数阶微分系统正解的唯一性。

$$\begin{cases} {}^cD_{0+}^{\alpha_1}\varphi_{p_1}(D_{0+}^{\beta_1}x(t))+\mu_1 f_1(t,y(t))+\mu_2 g_1(t,y(t))=0,0<t<1; \\ {}^cD_{0+}^{\alpha_2}\varphi_{p_2}(D_{0+}^{\beta_2}y(t))+\mu_1 f_2(t,x(t))+\mu_2 g_2(t,x(t))=0,0<t<1; \\ x^{(j)}(0)=0,j=0,1,2,\cdots,n-2,\ D_{0+}^{\beta_1}x(0)=0; \\ y^{(j)}(0)=0,j=0,1,2,\cdots,n-2,\ D_{0+}^{\beta_2}y(0)=0; \\ D_{0+}^{k_1}x(1)=\lambda_{11}\int_0^1 l_{11}(\tau)x(\tau)\mathrm{d}A_{11}(\tau)+\lambda_{12}\int_0^{\xi_1} l_{12}(\tau)x(\tau)\mathrm{d}A_{12}(\tau)+\lambda_{13}\sum_{i=1}^{\infty}\sigma_{1i}x(\eta_{1i})+\omega_1; \\ D_{0+}^{k_2}y(1)=\lambda_{21}\int_0^1 l_{21}(\tau)y(\tau)\mathrm{d}A_{21}(\tau)+\lambda_{22}\int_0^{\xi_2} l_{22}(\tau)Y(\tau)\mathrm{d}A_{22}(\tau)+\lambda_{23}\sum_{i=1}^{\infty}\sigma_{2i}y(\eta_{2i})+\omega_2。 \end{cases} \tag{1}$$

式中：$\alpha_i\in(0,1]$；$0<k_i<n-1<\beta_i\leqslant n,n\geqslant 3,\beta_i-k_i-1\geqslant 0$；$\mu_i,\omega_i\geqslant 0$ 为参数($i=1,2$)，且 $\mu_1\geqslant\mu_2,\lambda_{ji}\geqslant 0(j=1,2;i=1,2,3)$；$l_{ji}\in C((0,1),\mathbf{R}^+)\cap L^1((0,1),\mathbf{R}^+)(i,j=1,2)$；$\xi_j,\eta_{ji}\in(0,1],\sigma_{ji}\geqslant 0(j=1,2;\ i=1,2,\cdots,\infty)$；$\int_0^1 l_{i1}(\tau)\mathrm{d}A_{i1}(\tau)$ 与 $\int_0^{\xi_i} l_{i2}(\tau)\mathrm{d}A_{i2}(\tau)(i=1,2)$ 为 Riemann-Stieltjes 积分。$\varphi_{p_i}(s)=|s|^{p_i-2}\cdot s,p_i>1,\varphi_{q_i}=\varphi_{p_i}^{-1}$，本研究令 $p_i\geqslant 2(i=1,2)$。

1 预备知识

定义 1[17] 连续函数 $y:(0,\infty)\to\mathbf{R}$ 的 $\alpha>0$ 阶 Riemann-Liouville 分数阶积分定义为：

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s。$$

定义 2[18] 连续函数 $y:(0,\infty)\to\mathbf{R}$ 的 $\alpha>0$ 阶 Riemann-Liouville 分数阶导数定义为：

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t(t-s)^{n-\alpha-1}y(s)\mathrm{d}s,n-1<\alpha<n。$$

定义 3[17] 连续函数 $y:(0,\infty)\to\mathbf{R}$ 的 $\alpha>0$ 阶 Caputo 分数阶导数定义为：

$${}^cD_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,n-1<\alpha<n。$$

式中，等式右端在 $(0,\infty)$ 上是逐点定义的。

引理 1[18] 设 $y,D_{0+}^{\alpha}y\in C(0,1)\cap L^1(0,1)$，则

$$I_{0+}^{\alpha}D_{0+}^{\alpha}y(t)=y(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},c_i\in\mathbf{R},i=1,2,\cdots,n;n-1<\alpha\leqslant n。$$

引理 2[17] 设 $y\in AC^n[0,1]$，则

$$I_{0+}^{\alpha}D_{0+}^{\alpha}y(t)=y(t)+c_0+c_1t+\cdots+c_{n-1}t^{n-1},c_i\in\mathbf{R},i=1,2,\cdots,n;n-1<\alpha\leqslant n。$$

引理 3 设 $F(t)\in C(0,1)\cap L^1(0,1)$，则边值问题

$$\begin{cases} D_{0+}^{\beta_1}x(t)+F(t)=0,0<t<1 \\ x^{(j)}(0)=0,j=0,1,2,\cdots,n-2 \\ D_{0+}^{k_1}x(1)=\lambda_{11}\int_0^1 l_{11}(\tau)x(\tau)\mathrm{d}A_{11}(\tau)+\lambda_{12}\int_0^{\xi_1} l_{12}(\tau)x(\tau)\mathrm{d}A_{12}(\tau)+\lambda_{13}\sum_{i=1}^{\infty}\sigma_{1i}x(\eta_{1i})+\omega_1 \end{cases} \tag{2}$$

有唯一解 $x(t)=\int_0^1 G_1(t,s)F(s)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1$。式中：

$$G_1(t,s)=\sum_{i=0}^{3}G_{1i}(t,s);$$

$$G_{10}(t,s)=\frac{1}{\Gamma(\beta_1)}\begin{cases}t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}-(t-s)^{\beta_1-1},0\leqslant s\leqslant t\leqslant 1;\\ t^{\beta_1-1}(1-s)^{\beta_1-k_1-1},0\leqslant t\leqslant s\leqslant 1;\end{cases}$$

$$G_{11}(t,s)=t^{\beta_1-1}\int_0^1 H_{11}(\tau,s)l_{11}(\tau)\mathrm{d}A_{11}(\tau);G_{12}(t,s)=t^{\beta_1-1}\int_0^{\xi_1}H_{12}(\tau,s)l_{12}(\tau)\mathrm{d}A_{12}(\tau);$$

$$G_{13}(t,s)=t^{\beta_1-1}\sum_{i=1}^{\infty}\sigma_{1i}H_{13}(\eta_{1i},s);$$

$$\underset{(i=1,2,3)}{H_{1i}(\tau,s)}=\frac{\lambda_{1i}}{\Delta_1\Gamma(\beta_1)}\begin{cases}\tau^{\beta_1-1}(1-s)^{\beta_1-k_1-1}-(\tau-s)^{\beta_1-1},0\leqslant s\leqslant \tau\leqslant 1;\\ \tau^{\beta_1-1}(1-s)^{\beta_1-k_1-1},0\leqslant \tau\leqslant s\leqslant 1;\end{cases}$$

$$\Delta_1=\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-k_1)}-\lambda_{11}\int_0^1 l_{11}(\tau)\tau^{\beta_1-1}\mathrm{d}A_{11}(\tau)-\lambda_{12}\int_0^{\xi_1}l_{12}(\tau)\tau^{\beta_1-1}\mathrm{d}A_{12}(\tau)-\lambda_{13}\sum_{i=1}^{\infty}\sigma_{1i}\eta_{1i}^{\beta_1-1}\neq 0。$$

证明：根据引理 1，边值问题(2)有解 $x(t)=-I_{0+}^{\beta_1}F(t)+c_1t^{\beta_1-1}+c_2t^{\beta_1-2}+\cdots+c_nt^{\beta_1-n}$，$c_i\in\mathbf{R}$，$i=1,2,\cdots,n$。由边界条件 $x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0$，可知 $c_2=\cdots=c_{n-1}=c_n=0$，则

$$x(t)=-I_{0+}^{\beta_1}F(t)+c_1t^{\beta_1-1}。\tag{3}$$

结合边值问题(2)最后一个边界条件，可得：$c_1=\frac{1}{\Delta_1}\left\{\int_0^1\frac{(1-s)^{\beta_1-k_1-1}}{\Gamma(\beta_1-k_1)}F(s)\mathrm{d}s-\lambda_{11}\int_0^1\left(\int_0^\tau\frac{(\tau-s)^{\beta_1-1}}{\Gamma(\beta_1)}F(s)\mathrm{d}s\right)\right.$ $l_{11}(\tau)\mathrm{d}A_{11}(\tau)-\lambda_{12}\int_0^{\xi_1}\left(\int_0^\tau\frac{(\tau-s)^{\beta_1-1}}{\Gamma(\beta_1)}F(s)\mathrm{d}s\right)\cdot l_{12}(\tau)\mathrm{d}A_{12}(\tau)-\lambda_{13}\sum_{i=1}^{\infty}\sigma_{1i}\int_0^{\eta_{1i}}\frac{(\eta_{1i}-s)^{\beta_1-1}}{\Gamma(\beta_1)}F(s)\mathrm{d}s+\left.\omega_1\right\}$。最后，将 c_1 代入式(3)中，则有结论成立。

引理 4 设 $K_1(t)\in C(0,1)\cap L^1(0,1)$，则边值问题

$$\begin{cases}{}^cD_{0+}^{\alpha_1}\varphi_{p_1}(D_{0+}^{\beta_1}x(t))+K_1(t)=0,\ 0<t<1\\ x^{(j)}(0)=0,\ j=0,1,2,\cdots,n-2,D_{0+}^{\beta_1}x(0)=0\\ D_{0+}^{k_1}x(1)=\lambda_{11}\int_0^1 l_{11}(\tau)x(\tau)\mathrm{d}A_{11}(\tau)+\lambda_{12}\int_0^{\xi_1}l_{12}(\tau)x(\tau)\mathrm{d}A_{12}(\tau)+\lambda_{13}\sum_{i=1}^{\infty}\sigma_{1i}x(\eta_{1i})+\omega_1\end{cases}\tag{4}$$

有唯一解 $x(t)=\int_0^1 G_1(t,s)\varphi_{q_1}(I_{0+}^{\alpha_1}K_1(s))\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1$。

证明：令 $Z(t)=D_{0+}^{\beta_1}x(t)$，则边值问题(4)可以拆成以下两个方程组：

$$\begin{cases}D_{0+}^{\beta_1}x(t)=Z(t),0<t<1;\\ x^{(j)}(0)=0,j=0,1,2,\cdots,n-2;\\ D_{0+}^{k_1}x(1)=\lambda_{11}\int_0^1 l_{11}(\tau)x(\tau)\mathrm{d}A_{11}(\tau)+\lambda_{12}\int_0^{\xi_1}l_{12}(\tau)x(\tau)\mathrm{d}A_{12}(\tau)+\lambda_{13}\sum_{i=1}^{\infty}\sigma_{1i}x(\eta_{1i})+\omega_1;\end{cases}\tag{5}$$

$$\begin{cases}{}^cD_{0+}^{\alpha_1}\varphi_{p_1}(Z(t))+K_1(t)=0,\ 0<t<1;\\ Z(0)=0。\end{cases}\tag{6}$$

由引理 3 可知边值问题(5)有解

$$x(t)=-\int_0^1 G_1(t,s)Z(s)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1。\tag{7}$$

由引理 2 可知边值问题(6)有解

$$Z(t)=-\varphi_{q_1}(I_{0+}^{\alpha_1}K_1(t))。\tag{8}$$

结合式(7)、(8)则可得出结论。

引理 5 设 $K_2(t)\in C(0,1)\cap L^1(0,1)$，则边值问题

$$\begin{cases} {}^{c}D_{0+}^{\alpha_2}\varphi_{p_2}(D_{0+}^{\beta_2}y(t))+K_2(t)=0,\ 0<t<1 \\ y^{(j)}(0)=0,\ j=0,1,2,\cdots,n-2,\ D_{0+}^{\beta_2}y(0)=0 \\ D_{0+}^{k_2}y(1)=\lambda_{21}\int_0^1 l_{21}(\tau)y(\tau)\mathrm{d}A_{21}(\tau)+\lambda_{22}\int_0^{\xi_1} l_{22}(\tau)y(\tau)\mathrm{d}A_{22}(\tau)+\lambda_{23}\sum_{i=1}^{\infty}\sigma_{2i}y(\eta_{2i})+\omega_2 \end{cases}$$

有解 $y(t)=\int_0^1 G_2(t,s)\varphi_{q_2}(I_{0+}^{\alpha_2}K_2(s))\mathrm{d}s+\frac{t^{\beta_2-1}}{\Delta_2}\omega_2$。式中:

$$G_2(t,s)=\sum_{i=0}^{3}G_{2i}(t,s);$$

$$G_{20}(t,s)=\frac{1}{\Gamma(\beta_2)}\begin{cases} t^{\beta_2-1}(1-s)^{\beta_2-k_2-1}-(t-s)^{\beta_2-1},0\leqslant s\leqslant t\leqslant 1; \\ t^{\beta_2-1}(1-s)^{\beta_2-k_2-1},0\leqslant t\leqslant s\leqslant 1; \end{cases}$$

$$G_{21}(t,s)=t^{\beta_2-1}\int_0^1 H_{21}(\tau,s)l_{21}(\tau)\mathrm{d}A_{21}(\tau);G_{22}(t,s)=t^{\beta_2-1}\int_0^{\xi_2}H_{22}(\tau,s)l_{22}(\tau)\mathrm{d}A_{22}(\tau);$$

$$G_{23}(t,s)=t^{\beta_2-1}\sum_{i=1}^{\infty}\sigma_{2i}H_{23}(\eta_{2i},s);$$

$$\underset{(i=1,2,3)}{H_{2i}(\tau,s)}=\frac{\lambda_{2i}}{\Delta_2\Gamma(\beta_2)}\begin{cases} \tau^{\beta_2-1}(1-s)^{\beta_2-k_2-1}-(\tau-s)^{\beta_2-1},0\leqslant s\leqslant \tau\leqslant 1; \\ \tau^{\beta_2-1}(1-s)^{\beta_2-k_2-1},0\leqslant \tau\leqslant s\leqslant 1; \end{cases}$$

$$\Delta_2=\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-k_2)}-\lambda_{21}\int_0^1 l_{21}(\tau)\tau^{\beta_2-1}\mathrm{d}A_{21}(\tau)-\lambda_{22}\int_0^{\xi_2}l_{22}(\tau)\tau^{\beta_2-1}\mathrm{d}A_{22}(\tau)-\lambda_{23}\sum_{i=1}^{\infty}\sigma_{2i}\eta_{2i}^{\beta_2-1}\neq 0。$$

证明:根据引理3和引理4的证明,很容易得出结论。

引理6 当 $\Delta_j>0,\int_0^1\tau^{\beta_j-1}l_{j1}(\tau)\mathrm{d}A_{j1}(\tau)\geqslant 0,\int_0^{\xi_j}\tau^{\beta_j-1}l_{j2}(\tau)\mathrm{d}A_{j2}(\tau)\geqslant 0,\sum_{i=1}^{\infty}\sigma_{ji}\eta_{ji}^{\beta_j-1}\geqslant 0(j=1,2)$,则格林函数 $G_1(t,s),G_2(t,s)$ 具有如下性质:

1) $G_j(t,s)\geqslant 0,t,s\in(0,1),j=1,2$;

2) $t^{\beta_j-1}d_j\psi_j(s)\leqslant G_j(t,s)\leqslant t^{\beta_j-1}d_j,j=1,2$。式中,$\psi_j(s)=(1-s)^{\beta_j-k_j-1}(1-(1-s)^{k_j}),d_j=d_{j0}+d_{j1}+d_{j2}+d_{j3},d_{j0}=\frac{1}{\Gamma(\beta_j)},d_{j1}=\frac{\lambda_{j1}}{\Delta_j\Gamma(\beta_j)}\int_0^1\tau^{\beta_j-1}l_{j1}(\tau)\mathrm{d}A_{j1}(\tau),d_{j2}=\frac{\lambda_{j2}}{\Delta_j\Gamma(\beta_j)}\int_0^{\xi_j}\tau^{\beta_j-1}l_{j2}(\tau)\mathrm{d}A_{j2}(\tau)$,$d_{j3}=\frac{\lambda_{j3}}{\Delta_j\Gamma(\beta_j)}\sum_{i=1}^{\infty}\sigma_{ji}\eta_{ji}^{\beta_j-1}$。

证明:根据引理3与引理5,当 $0\leqslant s\leqslant t\leqslant 1$ 时,则有

$$G_{10}(t,s)=\frac{1}{\Gamma(\beta_1)}(t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}-(t-s)^{\beta_1-1})=\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}\left((1-s)^{\beta_1-k_1-1}-\left(1-\frac{s}{t}\right)^{\beta_1-1}\right)$$
$$\geqslant\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}((1-s)^{\beta_1-k_1-1}-(1-s)^{\beta_1-1})=\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}(1-(1-s)^{k_1}),$$
$$G_{10}(t,s)=\frac{1}{\Gamma(\beta_1)}(t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}-(t-s)^{\beta_1-1})\leqslant\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}\leqslant\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}。$$

当 $0\leqslant t\leqslant s\leqslant 1$ 时,$G_{10}(t,s)=\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}\geqslant\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}(1-(1-s)^{k_1})$,$G_{10}(t,s)=\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}\leqslant\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}$,即

$$\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}(1-s)^{\beta_1-k_1-1}(1-(1-s)^{k_1})\leqslant G_{10}(t,s)\leqslant\frac{1}{\Gamma(\beta_1)}t^{\beta_1-1}。\tag{9}$$

同理可得:

$$\frac{\lambda_{11}t^{\beta_1-1}}{\Delta_1\Gamma(\beta_1)}\int_0^1\tau^{\beta_1-1}l_{11}(\tau)\mathrm{d}A_{11}(\tau)(1-s)^{\beta_1-k_1-1}(1-(1-s)^{k_1})\leqslant G_{11}(t,s)\leqslant\frac{\lambda_{11}t^{\beta_1-1}}{\Delta_1\Gamma(\beta_1)}\int_0^1\tau^{\beta_1-1}l_{11}(\tau)\mathrm{d}A_{11}(\tau),\tag{10}$$

$$\frac{\lambda_{12}t^{\beta_1-1}}{\Delta_1\Gamma(\beta_1)}\int_0^{\xi_1}\tau^{\beta_1-1}l_{12}(\tau)\mathrm{d}A_{12}(\tau)(1-s)^{\beta_1-k_1-1}(1-(1-s)^{k_1})\leqslant G_{12}(t,s)\leqslant\frac{\lambda_{12}t^{\beta_1-1}}{\Delta_1\Gamma(\beta_1)}\int_0^{\xi_1}\tau^{\beta_1-1}l_{12}(\tau)\mathrm{d}A_{12}(\tau),\tag{11}$$

$$t^{\beta_1-1}\frac{\lambda_{13}}{\Delta_1\Gamma(\beta_1)}\sum_{i=1}^{\infty}\sigma_{1i}\eta_{1i}^{\beta_1-1}(1-s)^{\beta_1-k_1-1}(1-(1-s)^{k_1})\leqslant G_{13}(t,s)\leqslant t^{\beta_1-1}\frac{\lambda_{13}}{\Delta_1\Gamma(\beta_1)}\sum_{i=1}^{\infty}\sigma_{1i}\eta_{1i}^{\beta_1-1}。\tag{12}$$

又由于 $\lambda_{ji}\geqslant 0(j=1,2;i=1,2,3)$,$(1-s)^{\beta_1-k_1-1}(1-(1-s)^{k_1})\geqslant 0$,再结合已知条件和式(9)～(12),则有 $G_1(t,s)\geqslant 0$。同理可得出 $G_2(t,s)\geqslant 0$,即性质(1)成立。同时可得:$t^{\beta_j-1}d_j\psi_j(s)\leqslant G_j(t,s)\leqslant t^{\beta_j-1}d_j$,$j=1,2$,即性质(2)成立。

定义 4[19] 设 E 为实 Banach 空间,对于锥 $P\subset E$,定义 E 中的偏序关系,即 $y-x\in P\Leftrightarrow x\leqslant y$。若存在一个常数 $N>0$,使得 $x,y\in E,\theta\leqslant x\leqslant y$ 满足 $\|x\|\leqslant N\|y\|$,则锥 P 是正规锥,其中 $\theta\in E$ 中的零元。若 $\mathring{P}$ 为非空集,则锥 P 是体锥。

定义 5[19] 若 $x\leqslant y$,有 $Tx\leqslant Ty(Tx\geqslant Ty)$,则称 $T:E\to E$ 为增算子(减算子)。

定义 6[19] 设 $0<\gamma<1$,若对 $\tau\in(0,1)$,$x\in P$,有 $T(\tau x)\geqslant\tau^{\gamma}Tx$,则称算子 $T:P\to P$ 是 γ-凹算子。若对 $\tau>0,x\in P$,有 $T(\tau x)\geqslant\tau T(x)$,则称算子 $T:P\to P$ 是次齐次的。对 $\forall x,y\in E,x\sim y$ 表示 $\exists\lambda>0,\mu>0$ 使得 $\lambda x\leqslant y\leqslant\mu x$,显然,$\sim$ 为等价关系。给定 $h>\theta(h\geqslant\theta$ 且 $h\neq\theta)$,定义集合 $P_h=\{x\in E\,|\,x\sim h\}$,易得 $P_h\subset P$。

引理 7[20] 设 P 是实 Banach 空间 E 中的正规锥,$A:P\to P$ 是增的 γ-凹算子,$B:P\to P$ 是增的次齐次算子。若满足条件:①存在 $h>\theta$,使得 $Ah\in P_h,Bh\in P_h$;②存在一个常数 $\delta_0>0$,使得 $Ax>\delta_0 Bx$,$x\in P$,则算子方程 $Ax+Bx=x$ 在 P_h 中有唯一解 x^*;且对于任意给定的初值 $y_0\in P_h$,做迭代序列 $y_n=Ay_{n-1}+By_{n-1},n=1,2,\cdots$,则 $\{y_n\}$ 收敛于 x^*。

引理 8[21] 设 P 是 E 中的正规锥,$A:P\to P$ 是增算子,$B:P\to P$ 是减算子。若下面条件成立:

1) 对 $t\in(0,1)$,$\exists\phi_i(t)\in(t,1)(i=1,2)$,使得

$$A(tx)\geqslant\phi_1(t)Ax,B(tx)\leqslant\frac{1}{\phi_2(t)}Bx,x\in P。\tag{13}$$

2) 存在 $h_0\in P_h$,使得 $Ah_0+Bh_0\in P_h$。则算子方程 $Ax+Bx=x$ 在 P_h 中有唯一解 x^*,且对于任意给定的初值 $(x_0,y_0)\in P_h$,做迭代序列 $x_n=Ax_{n-1}+By_{n-1},y_n=Ay_{n-1}+Bx_{n-1},n=1,2,\cdots$,则 $\{x_n\}$,$\{y_n\}$ 收敛于 x^*。

2 主要结果

设 Banach 空间 $X=C[0,1]$,其范数为 $\|x\|_X=\sup_{t\in[0,1]}|x(t)|$。对于 $(x,y)\in X\times X$,令 $E=X\times X$,设 $\|(x,y)\|_E=\max\{\|x\|_X,\|y\|_X\}$,显然,$(E,\|(x,y)\|_E)$ 是 Banach 空间。定义 E 中的正规锥 P 为:$P=\{(x,y)\in E\mid x(t)\geqslant 0,y(t)\geqslant 0,t\in[0,1]\}$。定义 E 中的偏序关系为:对任意 $t\in[0,1]$,有 $x,y\in C[0,1],x\leqslant y\Leftrightarrow x(t)\leqslant y(t)$。

根据引理 4 与引理 5,可知边值问题(1)等价于积分方程

$$\begin{cases}x(t)=\int_0^1 G_1(t,s)\varphi_{q_1}[I_{0+}^{\alpha_1}\mu_1 f_1(s,y(s))]\mathrm{d}s+\int_0^1 G_1(t,s)\varphi_{q_1}[I_{0+}^{\alpha_1}\mu_2 g_1(s,y(s))]\mathrm{d}s+\dfrac{t^{\beta_1-1}}{\Delta_1}\omega_1,\\ y(t)=\int_0^1 G_2(t,s)\varphi_{q_2}[I_{0+}^{\alpha_2}\mu_1 f_2(s,x(s))]\mathrm{d}s+\int_0^1 G_2(t,s)\varphi_{q_2}[I_{0+}^{\alpha_2}\mu_2 g_2(s,x(s))]\mathrm{d}s+\dfrac{t^{\beta_2-1}}{\Delta_2}\omega_2。\end{cases}$$

定义 4 个算子 $A_1,A_2,B_1,B_2:X\to X$,分别为:$A_1y(t)=\int_0^1 G_1(t,s)\varphi_{q_1}(I_{0+}^{\alpha_1}\mu_1 f_1(s,y(s)))\mathrm{d}s$,$A_2x(t)=\int_0^1 G_2(t,s)\varphi_{q_2}[I_{0+}^{\alpha_2}\mu_1 f_2(s,x(s))]\mathrm{d}s$,$B_1y(t)=\int_0^1 G_1(t,s)\varphi_{q_1}[I_{0+}^{\alpha_1}\mu_2 g_1(s,y(s))]\mathrm{d}s+\dfrac{t^{\beta_1-1}}{\Delta_1}\omega_1$,$B_2x(t)=\int_0^1 G_2(t,s)\varphi_{q_2}[I_{0+}^{\alpha_2}\mu_2 g_2(s,x(s))]\mathrm{d}s+\dfrac{t^{\beta_2-1}}{\Delta_2}\omega_2$。

定义 E 上两个算子 A,B：$A(x,y)=(A_1y,A_2x)$，$B(x,y)=(B_1y,B_2x)$。显然，算子 $A+B$ 的不动点 (x,y) 是边值问题(1)的解。

提出如下假设：

H_1）$f_i:C([0,1]\times\mathbf{R}^+,\mathbf{R}^+)$ 与 $g_i:C([0,1]\times\mathbf{R}^+,\mathbf{R}^+)$ 均在第 2 个变量为递增，且 $g_i(t,0)\neq 0$，$(i=1,2)$。

H_2）对 $\lambda\in(0,1)$，$t\in[0,1]$，$(x,y)\in P$，有 $g_1(t,\lambda y)\geqslant\lambda^{p_1-1}g_1(t,y)$，$g_2(t,\lambda x)\geqslant\lambda^{p_2-1}g_2(t,x)$；存在一个常数 $\gamma\in(0,1)$，对于 $\lambda\in(0,1)$，$x,y\in\mathbf{R}^+$，$t\in[0,1]$，使得 $f_1(t,\lambda y)\geqslant(\lambda^\gamma)^{p_1-1}f_1(t,y)$，$f_2(t,\lambda x)\geqslant(\lambda^\gamma)^{p_2-1}f_2(t,x)$。

H_3）对 $t\in[0,1]$，$(x,y)\in P$，存在常数 $\delta_0>0$，有 $f_1(t,y)\geqslant\delta_0^{p_1-1}g_1(t,y)+\dfrac{\Gamma(\alpha_1+1)}{\mu_2}\left(\dfrac{\delta_0^2(\alpha_1(q_1-1)+1)}{\Delta_1\psi_1(s)d_1}\right)^{p_1-1}$，$f_2(t,x)\geqslant\delta_0^{p_2-1}g_2(t,x)+\dfrac{\Gamma(\alpha_2+1)}{\mu_2}\left(\dfrac{\delta_0^2(\alpha_2(q_2-1)+1)}{\Delta_2\psi_2(s)d_2}\right)^{p_2-1}$。

H_4）$f_i:C([0,1]\times\mathbf{R}^+,\mathbf{R}^+)$ 在第 2 个变量为递增，且 $f_i(t,0)\neq 0(i=1,2)$。

H_5）存在 $\gamma\in(0,1)$，使得 $f_1(t,\lambda y)\geqslant(\lambda^\gamma)^{p_1-1}f_1(t,y)$，$f_2(t,\lambda x)\geqslant(\lambda^\gamma)^{p_2-1}f_2(t,x)$，$\lambda\in(0,1)$，$(x,y)\in P$，$t\in[0,1]$ 成立。

H_6）$f_i:C([0,1]\times\mathbf{R}^+,\mathbf{R}^+)$ 在第 2 个变量为递增，$g_i:C([0,1]\times\mathbf{R}^+,\mathbf{R}^+)$ 在第 2 个变量为递减，且对 $t\in[0,1]$ 有 $f_i(t,0)\neq 0$，$g_i(t,1)\neq 0$，$i=1,2$。

H_7）对于 $\varepsilon\in(0,1)$，存在 $\phi_i(\varepsilon)\in(\varepsilon,1)$ $(i=1,2)$ 使得 $f_1(t,\varepsilon y)\geqslant\phi_1^{p_1-1}(\varepsilon)f_1(t,y)$，$f_2(t,\varepsilon x)\geqslant\phi_1^{p_2-1}(\varepsilon)f_2(t,x)$；$g_1(t,\varepsilon y)\leqslant\left(\dfrac{1}{\phi_2(\varepsilon)}\right)^{p_1-1}g_1(t,y)$，$g_2(t,\varepsilon x)\leqslant\left(\dfrac{1}{\phi_2(\varepsilon)}\right)^{p_2-1}g_2(t,x)$，$t\in[0,1]$，$x,y\in\mathbf{R}^+$。

定理 1 若假设 H_1）～H_3）成立，则对每一个 $\omega_1,\omega_2\in(0,\delta_0]$，边值问题(1)有唯一正解 $(x^*,y^*)\in P_h$，其中 $h(t)=(h_1(t),h_2(t))=(t^{\beta_1-1},t^{\beta_2-1})$。此外，对于任意初值 $(x_0,y_0)\in P_h$ 构造迭代序列

$$\begin{cases}x_n(t)=\displaystyle\int_0^1G_1(t,s)\varphi_{q_1}(I_{0+}^{\alpha_1}(\mu_1f_1(s,y_{n-1}(s))+\mu_2g_1(s,y_{n-1}(s))))\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1,\\ y_n(t)=\displaystyle\int_0^1G_2(t,s)\varphi_{q_2}(I_{0+}^{\alpha_2}(\mu_1f_2(s,x_{n-1}(s))+\mu_2g_2(s,x_{n-1}(s))))\mathrm{d}s+\frac{t^{\beta_2-1}}{\Delta_2}\omega_2,\end{cases}$$

$n=1,2,\cdots$，并且当 $n\to\infty$ 时，有 $(x_n(t),y_n(t))\to(x^*(t),y^*(t))$。

证明：由假设 H_1)与引理 6 很容易得出 $A:P\to P$，$B:P\to P$。

设 (x,y)，$(u,v)\in P$，且 $x\geqslant u$，$y\geqslant v$，由假设 H_1)可得：

$$A_1y(t)=\int_0^1G_1(t,s)\varphi_{q_1}(I_{0+}^{\alpha_1}\mu_1f_1(s,y(s)))\mathrm{d}s\geqslant\int_0^1G_1(t,s)\varphi_{q_1}(I_{0+}^{\alpha_1}\mu_1f_1(s,v(s)))\mathrm{d}s=A_1v(t),$$

$$A_2x(t)=\int_0^1G_2(t,s)\varphi_{q_2}(I_{0+}^{\alpha_2}\mu_1f_2(s,x(s)))\mathrm{d}s\geqslant\int_0^1G_2(t,s)\varphi_{q_2}(I_{0+}^{\alpha_2}\mu_1f_2(s,u(s)))\mathrm{d}s=A_2u(t),$$

从而 $A(x,y)\geqslant A(u,v)$。同理可得 $B(x,y)\geqslant B(u,v)$。因此 $A:P\to P$，$B:P\to P$ 都是增算子。

对于 $\lambda\in(0,1)$，$\gamma\in(0,1)$，$(x,y)\in P$，$t\in[0,1]$，由假设 H_2)可得：

$$A_1(\lambda y)(t)=\int_0^1G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1f_1(\tau,\lambda y(\tau))\mathrm{d}\tau\right)\mathrm{d}s\geqslant$$

$$\int_0^1G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1(\lambda^\gamma)^{p_1-1}f_1(\tau,y(\tau))\mathrm{d}\tau\right)\mathrm{d}s=\lambda^\gamma A_1y(t),$$

$$A_2(\lambda x)(t)=\int_0^1G_2(t,s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1f_2(\tau,\lambda x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\geqslant$$

$$\int_0^1G_2(t,s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1(\lambda^\gamma)^{p_2-1}f_2(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s=\lambda^\gamma A_2x(t),$$

即 $A(\lambda(x,y))\geqslant\lambda^\gamma A(x,y)$ 成立。因此 A 是一个 γ-凹算子。

对于 $\lambda \in (0,1)$，$(x,y) \in P$，$t \in [0,1]$，由假设条件 H_2)有

$$B_1(\lambda y)(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,\lambda y(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1 \geqslant$$

$$\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 \lambda^{p_1-1} g_1(\tau,y(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\lambda\frac{t^{\beta_1-1}}{\Delta_1}\omega_1=\lambda B_1 y(t),$$

$$B_2(\lambda x)(t)=\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,\lambda x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_2-1}}{\Delta_2}\omega_2 \geqslant$$

$$\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 \lambda^{p_2-1} g_2(\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\lambda\frac{t^{\beta_2-1}}{\Delta_2}\omega_2=\lambda B_2 x(t),$$

即 $B(\lambda(x,y)) \geqslant \lambda B(x,y)$ 成立。因此 B 是一个次齐次算子。

结合假设条件 H_1)和引理 6 有

$$A_1 h_2(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1 f_1(\tau,h_2(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \leqslant$$

$$h_1(t)\cdot d_1\int_0^1 \varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1 f_1(\tau,1)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$A_2 h_1(t)=\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1 f_2(\tau,h_1(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \leqslant$$

$$h_2(t)\cdot d_2\int_0^1 \varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1 f_2(\tau,1)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$A_1 h_2(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1 f_1(\tau,h_2(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \geqslant$$

$$h_1(t)\cdot d_1\int_0^1 \psi_1(s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1 f_1(\tau,0)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$A_2 h_1(t)=\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1 f_2(\tau,h_1(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \geqslant$$

$$h_2(t)\cdot d_2\int_0^1 \psi_2(s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1 f_2(\tau,0)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$B_1 h_2(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,h_2(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1 \leqslant$$

$$h_1(t)\cdot\left\{d_1\int_0^1 \varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,1)\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_1}{\Delta_1}\right\},$$

$$B_2 h_1(t)=\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,h_1(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_2-1}}{\Delta_2}\omega_2 \leqslant$$

$$h_2(t)\cdot\left\{d_2\int_0^1 \varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,1)\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_2}{\Delta_2}\right\},$$

$$B_1 h_2(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,h_2(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1 \geqslant$$

$$h_1(t)\cdot d_1\int_0^1 \psi_1(s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,0)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$B_2 h_1(t)=\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,h_1(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_2-1}}{\Delta_2}\omega_2 \geqslant$$

$$h_2(t)\cdot d_2\int_0^1 \psi_2(s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,0)\,\mathrm{d}\tau\right)\mathrm{d}s。$$

令 $b=\min\left\{d_1\int_0^1 \varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1 f_1(\tau,1)\,\mathrm{d}\tau\right)\mathrm{d}s, d_2\int_0^1 \varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1 f_2(\tau,1)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}$，$a=$

$\max\left\{d_1\int_0^1\psi_1(s)\cdot\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1 f_1(\tau,0)\mathrm{d}\tau\right)\mathrm{d}s, d_2\int_0^1\psi_2(s)\cdot\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1 f_2(\tau,0)\mathrm{d}\tau\right)\mathrm{d}s\right\}$,

$m=\min\left\{d_1\int_0^1\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,1)\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_1}{\Delta_1}, d_2\int_0^1\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,1)\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_2}{\Delta_2}\right\}$,

$l=\max\left\{d_1\int_0^1\psi_1(s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,0)\mathrm{d}\tau\right)\mathrm{d}s, d_2\int_0^1\psi_2(s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,0)\mathrm{d}\tau\right)\mathrm{d}s\right\}$,

又结合假设条件 H_1)、H_3)可知,$f_i(t,1)\geqslant f_i(t,0)\geqslant\delta_0^{p_i-1}g_i(t,0)>0$,$(i=1,2)$;又由于 $\psi_i(s)=(1-s)^{\beta_i-k_i-1}\cdot(1-(1-s)^{k_i})\leqslant 1$,则 $0<a\leqslant b$,$0<l\leqslant m$,从而 $ah\leqslant Ah\leqslant bh$,$lh\leqslant Bh\leqslant mh$,因此 $Ah\in P_h$,$Bh\in P_h$。

对于 $(x,y)\in P$,由假设条件 H_3)有

$$A_1y(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1 f_1(\tau,y(\tau))\mathrm{d}\tau\right)\mathrm{d}s\geqslant$$

$$\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2\left(\delta_0^{p_1-1}g_1(\tau,y(\tau))+\frac{\Gamma(\alpha_1+1)}{\mu_2}\left(\frac{\delta_0^2(\alpha_1(q_1-1)+1)}{\Delta_1\psi_1(s)d_1}\right)^{p_1-1}\right)\mathrm{d}\tau\right)\mathrm{d}s\geqslant$$

$$\delta_0\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,y(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\int_0^1 t^{\beta_1-1}d_1\psi_1(s)\cdot\frac{\delta_0^2(\alpha_1(q_1-1)+1)}{\Delta_1\psi_1(s)d_1}\cdot\varphi_{q_1}(s^{\alpha_1})\mathrm{d}s\geqslant$$

$$\delta_0\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2 g_1(\tau,y(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\delta_0\cdot\frac{t^{\beta_1-1}}{\Delta_1}\omega_1=\delta_0B_1y(t),$$

$$A_2x(t)=\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1 f_2(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\geqslant$$

$$\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2\left(\delta_0^{p_2-1}g_2(\tau,x(\tau))+\frac{\Gamma(\alpha_2+1)}{\mu_2}\left(\frac{\delta_0^2(\alpha_2(q_2-1)+1)}{\Delta_2\psi_2(s)d_2}\right)^{p_2-1}\right)\mathrm{d}\tau\right)\mathrm{d}s\geqslant$$

$$\delta_0\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\int_0^1 t^{\beta_2-1}d_2\psi_2(s)\cdot\frac{\delta_0^2(\alpha_2(q_2-1)+1)}{\Delta_2\psi_2(s)d_2}\cdot\varphi_{q_2}(s^{\alpha_2})\mathrm{d}s\geqslant$$

$\delta_0\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2 g_2(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\delta_0\cdot\frac{t^{\beta_2-1}}{\Delta_2}\omega_2=\delta_0B_2x(t)$,即对 $(x,y)\in P$,有 $A_1y(t)\geqslant\delta_0B_1y(t)$,$A_2x(t)\geqslant\delta_0B_2x(t)$。因此 $\forall(x,y)\in P$,有 $A(x,y)\geqslant\delta_0B(x,y)$ 成立。

最后,由引理 7 可知,算子方程 $(x,y)=A(x,y)+B(x,y)$ 有唯一正解 $(x^*,y^*)\in P_h$。且对任意初值 $(x_0,y_0)\in P_h$,构造迭代序列 $(x_n,y_n)=A(x_{n-1},y_{n-1})+B(x_{n-1},y_{n-1})$,$n=1,2,\cdots$,当 $n\to\infty$ 时,$(x_n,y_n)\to(x^*,y^*)$。所以边值问题(1)有唯一正解 $(x^*,y^*)\in P_h$,且有迭代序列

$$\begin{cases}x_n(t)=\int_0^1 G_1(t,s)\varphi_{q_1}[I_{0+}^{\alpha_1}[\mu_1 f_1(s,y_{n-1}(s))+\mu_2 g_1(s,y_{n-1}(s))]]\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1,\\ y_n(t)=\int_0^1 G_2(t,s)\varphi_{q_2}[I_{0+}^{\alpha_2}[\mu_1 f_2(s,x_{n-1}(s))+\mu_2 g_2(s,x_{n-1}(s))]]\mathrm{d}s+\frac{t^{\beta_2-1}}{\Delta_2}\omega_2,\end{cases}$$

$n=1,2,\cdots$,对于任意初值 $(x_0,y_0)\in P_h$,当 $n\to\infty$ 时,有 $(x_n(t),y_n(t))\to(x^*(t),y^*(t))$。所以定理 1 成立。

推论 1 当边值问题(1)中的函数 $g_i(t,s)=0$ 时,若假设条件 H_4)、H_5)成立,则此时的边值问题(1)对于每一个 $\omega_1,\omega_2\in(0,\delta_0]$,有唯一正解 $(x^*,y^*)\in P_h$,其中 $h(t)=(h_1(t),h_2(t))=(t^{\beta_1-1},t^{\beta_2-1})$。此外,对于任意初值 $(x_0,y_0)\in P_h$ 构造迭代序列

$$(x_n(t),y_n(t))=\left(\int_0^1 G_1(t,s)\varphi_{q_1}(I_{0+}^{\alpha_1}(\mu_1 f_1(s,y_{n-1}(s))))\mathrm{d}s+\frac{\omega_1t^{\beta_1-1}}{\Delta_1},\right.$$

$\left.\int_0^1 G_2(t,s)\varphi_{q_2}(I_{0+}^{\alpha_2}(\mu_1 f_2(s,x_{n-1}(s))))\mathrm{d}s+\frac{\omega_2t^{\beta_2-1}}{\Delta_2}\right)$,$n=1,2,\cdots$。当 $n\to\infty$ 时,$(x_n(t),y_n(t))\to(x^*(t),y^*(t))$。

定理2 若假设条件 H_6)～H_7)成立,则边值问题(1)有唯一正解 $(x^*,y^*)\in P_h$,其中 $h(t)=(h_1(t),h_2(t))=(t^{\beta_1-1},t^{\beta_2-1})$。且对任意给定初值 $(x_0(t),y_0(t)),(u_0(t),v_0(t))\in P_h$ 做迭代序列

$$\begin{cases}(x_n(t),y_n(t))=\left(\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}(\mu_1 f_1(\tau,y_{n-1}(\tau))+\mu_2 g_1(\tau,v_{n-1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_1 t^{\beta_1-1}}{\Delta_1},\right.\\ \left.\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}(\mu_1 f_2(\tau,x_{n-1}(\tau))+\mu_2 g_2(\tau,u_{n-1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_2 t^{\beta_2-1}}{\Delta_2}\right),\\ (u_n(t),v_n(t))=\left(\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}(\mu_1 f_1(\tau,v_{n-1}(\tau))+\mu_2 g_1(\tau,y_{n-1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_1 t^{\beta_1-1}}{\Delta_1},\right.\\ \left.\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^s \frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}(\mu_1 f_2(\tau,u_{n-1}(\tau))+\mu_2 g_2(\tau,x_{n-1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_2 t^{\beta_2-1}}{\Delta_2}\right),\end{cases}$$

$n=1,2,\cdots$。当 $n\to\infty$ 时,有 $(x_n(t),y_n(t))\to(x^*(t),y^*(t)),(u_n(t),v_n(t))\to(x^*(t),y^*(t))$。

证明:由假设条件 H_6)很容易得出 $A:P\to P$ 为增算子,$B:P\to P$ 为减算子。由假设条件 H_7)很容易得出算子 A,B 满足式(13)。记

$$M=\min\left\{d_1\int_0^1\psi_1(s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2(f_1(\tau,0)+g_1(\tau,1))\mathrm{d}\tau\right)\mathrm{d}s,\right.$$
$$\left.d_2\int_0^1\psi_2(s)\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2(f_2(\tau,0)+g_2(\tau,1))\mathrm{d}\tau\right)\mathrm{d}s\right\},$$
$$N=\max\left\{d_1\int_0^1\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1(f_1(\tau,1)+g_1(\tau,0))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_1}{\Delta_1},\right.$$
$$\left.d_2\int_0^1\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1(f_2(\tau,1)+g_2(\tau,0))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\omega_2}{\Delta_2}\right\}。$$

由假设条件 H_6)与引理6,可得:

$A_1h_2(t)+B_1h_2(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}(\mu_1 f_1(\tau,h_2(\tau))+\mu_2 g_1(\tau,h_2(\tau)))\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1\leqslant\int_0^1 t^{\beta_1-1}d_1\cdot\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_1(f_1(\tau,1)+g_1(\tau,0))\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1\leqslant Nh_1(t)$,$A_1h_2(t)+B_1h_2(t)=\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}(\mu_1 f_1(\tau,h_2(\tau))+\mu_2 g_1(\tau,h_2(\tau)))\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_1-1}}{\Delta_1}\omega_1\geqslant\int_0^1 t^{\beta_1-1}d_1\psi_1(s)\cdot\varphi_{q_1}\left(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mu_2(f_1(\tau,0)+g_1(\tau,1))\mathrm{d}\tau\right)\mathrm{d}s\geqslant Mh_1(t)$。

同理可得:

$A_2h_1(t)+B_2h_1(t)\leqslant\int_0^1 t^{\beta_2-1}d_2\cdot\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_1(f_2(\tau,1)+g_2(\tau,0))\mathrm{d}\tau\right)\mathrm{d}s+\frac{t^{\beta_2-1}}{\Delta_2}\omega_2\leqslant Nh_2(t)$,$A_2h_1(t)+B_2h_1(t)\geqslant\int_0^1 t^{\beta_2-1}d_2\psi_2(s)\cdot\varphi_{q_2}\left(\int_0^s\frac{(s-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}\mu_2(f_2(\tau,0)+g_2(\tau,1))\mathrm{d}\tau\right)\mathrm{d}s\geqslant Mh_2(t)$。

由假设条件 H_6)可知,$0<f_i(t,0)\leqslant f_i(t,1)$,$0<g_i(t,1)\leqslant g_i(t,0)$,故 $0<M\leqslant N$,从而有 $Mh(t)\leqslant Ah(t)+Bh(t)\leqslant Nh(t)$,$t\in[0,1]$,即 $Ah(t)+Bh(t)\in P_h$。

故引理8中所有条件都成立,从而定理2成立。

3 数值例子

考虑以下分数阶微分系统

$$\begin{cases} {}^cD_{0+}^{1/3}\varphi_2(D_{0+}^{41/14}x(t))+3f_1(t,y(t))+3g_1(t,y(t))=0,\ 0<t<1;\\ {}^cD_{0+}^{1/5}\varphi_3(D_{0+}^{50/17}y(t))+3f_2(t,x(t))+3g_2(t,x(t))=0,\ 0<t<1;\\ x(0)=x'(0)=D_{0+}^{41/14}x(0)=0;\\ y(0)=y'(0)=D_{0+}^{50/17}y(0)=0;\\ D_{0+}^{13/14}x(1)=\frac{1}{10}\int_0^1\tau^{-27/14}x(\tau)\mathrm{d}A_{11}(\tau)+\frac{1}{10}\int_0^{7/8}\tau^{-27/14}x(\tau)\mathrm{d}A_{12}(\tau)+\frac{1}{10}\sum_{i=1}^{\infty}\frac{2}{i(i+1)}x(1)+\omega_1;\\ D_{0+}^{16/17}y(1)=\frac{1}{10}\int_0^1\tau^{-16/17}y(\tau)\mathrm{d}A_{21}(\tau)+\frac{1}{20}\int_0^{10/11}\tau^{-33/17}y(\tau)\mathrm{d}A_{22}(\tau)+\frac{3}{10}\sum_{i=1}^{\infty}3^{1+i\cdot 16/17}Y(3^{-i})+\omega_2。\end{cases} \tag{14}$$

式中:$\alpha_1=\frac{1}{3},\alpha_2=\frac{1}{5};p_1=2,p_2=3;q_1=2,q_2=\frac{3}{2};\beta_1=\frac{41}{14},\beta_2=\frac{50}{17}(n=3);\mu_1=\mu_2=3;k_1=\frac{13}{14},k_2=\frac{16}{17};$ $\lambda_{11}=\lambda_{12}=\lambda_{13}=\lambda_{21}=\frac{1}{10},\lambda_{22}=\frac{1}{20},\lambda_{23}=\frac{3}{10};l_{11}(t)=l_{12}(t)=t^{-\frac{27}{14}},l_{21}(t)=t^{-\frac{16}{17}},l_{22}(t)=t^{-\frac{33}{17}};\eta_{1i}=1,\eta_{2i}=3^{-i};$ $\sigma_{1i}=\frac{2}{i(i+1)},\sigma_{2i}=3^{1+\frac{16}{17}i};\xi_1=\frac{7}{8},\xi_2=\frac{10}{11};\omega_1=\frac{1}{20},\omega_2=0.08$。$f_1(t,y)=\frac{y^{\frac{1}{5}}}{1+t^2}+r+\frac{2Q\Gamma\left(\frac{4}{3}\right)}{405},g_1(t,y)=\frac{y}{(1+y)(1+t^{\frac{1}{2}})}+e,f_2(t,x)=\frac{x^{\frac{1}{3}}}{3+t^4}+r+\frac{121Q^2\Gamma\left(\frac{6}{5}\right)}{2\ 484\ 300},g_2(t,x)=\frac{x^{\frac{1}{2}}}{(1+x^{\frac{1}{2}})(4+t^2)}+e$。式中 $0\leqslant e\leqslant r$,Q 为一个 $Q\to\infty$ 的常数,同时令 $\min_{s\in[0,1]}\psi_i(s)=\frac{1}{Q}(i=1,2)$。

$$A_{11}(t)=\begin{cases}\frac{7}{16},t\in\left[0,\frac{1}{2}\right);\\ \frac{15}{16},t\in\left[\frac{1}{2},1\right];\end{cases}\quad A_{12}(t)=\begin{cases}\frac{1}{12},t\in[0,1);\\ \frac{13}{12},t=1;\end{cases}$$

$$A_{21}(t)=\begin{cases}\frac{1}{14},t\in\left[0,\frac{1}{2}\right);\\ \frac{15}{14},t\in\left[\frac{1}{2},1\right];\end{cases}\quad A_{22}(t)=\begin{cases}\frac{5}{18},t\in[0,1);\\ \frac{23}{18},t=1;\end{cases}$$

算得:$\int_0^1\tau^{\beta_1-1}l_{11}(\tau)\,\mathrm{d}A_{11}(\tau)=\frac{1}{2},\int_0^{\xi_1}\tau^{\beta_1-1}l_{12}(\tau)\,\mathrm{d}A_{12}(\tau)=1,\sum_{i=1}^{\infty}\sigma_{1i}\eta_{1i}^{\beta_1-1}=2,\Delta_1=1.524\ 3>0,d_1\approx 0.656\ 0;$ $\int_0^1\tau^{\beta_2-1}l_{21}(\tau)\,\mathrm{d}A_{21}(\tau)=\frac{1}{2},\int_0^{\xi_2}\tau^{\beta_2-1}l_{22}(\tau)\,\mathrm{d}A_{22}(\tau)=1,\sum_{i=1}^{\infty}\sigma_{2i}\eta_{2i}^{\beta_2-1}=\frac{3}{2},\Delta_2=1.345\ 6>0,d_2\approx 0.743\ 1$。$g_1(t,0)=e\geqslant 0,g_2(t,0)=e\geqslant 0$。由 f_i,g_i 的表达式可知 f_i,g_i 为连续函数,且在第 2 个变量为递增,所以 H_1)成立。

令 $\gamma=\frac{1}{3}$,对 $\forall\lambda\in(0,1),x,y\in[0,\infty),t\in[0,1]$ 有 $f_1(t,\lambda y)=\frac{(\lambda y)^{\frac{1}{5}}}{1+t^2}+r+\frac{2Q\Gamma\left(\frac{4}{3}\right)}{405}\geqslant\lambda^{\frac{1}{5}}\left[\frac{y^{\frac{1}{5}}}{1+t^2}+r+\frac{2Q\Gamma\left(\frac{4}{3}\right)}{405}\right]\geqslant\lambda^{\frac{1}{3}}\left[\frac{y^{\frac{1}{5}}}{1+t^2}+r+\frac{2Q\Gamma\left(\frac{4}{3}\right)}{405}\right]=\lambda^{\frac{1}{3}}f_1(t,y)$,$f_2(t,\lambda x)=\frac{(\lambda x)^{\frac{1}{3}}}{3+t^4}+r+\frac{121Q^2\Gamma\left(\frac{6}{5}\right)}{2\ 484\ 300}\geqslant\lambda^{\frac{1}{3}}\left[\frac{x^{\frac{1}{3}}}{3+t^4}+r+\frac{121Q^2\Gamma\left(\frac{6}{5}\right)}{2\ 484\ 300}\right]\geqslant(\lambda^{\frac{1}{3}})^2f_2(t,x)$,$g_1(t,\lambda y)=\frac{\lambda y}{(1+\lambda y)(1+t^{\frac{1}{2}})}+e\geqslant\lambda g_1(t,y)$,$g_2(t,\lambda x)=\frac{(\lambda x)^{\frac{1}{2}}}{(1+(\lambda x)^{\frac{1}{2}})(4+t^2)}+e\geqslant\lambda g_2(t,x)$,所以 H_2)成立。

令 $\delta_0=\dfrac{1}{10}$，则有 $\delta_0^{p_1-1}g_1(t,y)+\dfrac{\Gamma(\alpha_1+1)}{\mu_2}\left(\dfrac{\delta_0^2(\alpha_1(q_1-1)+1)}{\Delta_1\psi_1(s)d_1}\right)^{p_1-1}=\dfrac{1}{10}\left[\dfrac{y}{(1+y)(1+t^{\frac{1}{2}})}+e\right]+\dfrac{\Gamma\left(\frac{4}{3}\right)\times\left(\frac{1}{10}\right)^2\left(\frac{1}{3}\times1+1\right)}{3\Delta_1\psi_1(s)d_1}\leqslant f_1(t,y)$，$\delta_0^{p_2-1}g_2(t,x)+\dfrac{\Gamma(\alpha_2+1)}{\mu_2}\left(\dfrac{\delta_0^2(\alpha_2(q_2-1)+1)}{\Delta_2\psi_2(s)d_2}\right)^{p_2-1}=\dfrac{1}{100}\left[\dfrac{x^{\frac{1}{2}}}{(1+x^{\frac{1}{2}})(4+t^2)}+r\right]+\dfrac{\Gamma\left(\frac{6}{5}\right)}{3}\cdot\left[\dfrac{\frac{1}{100}\times\left(\frac{1}{5}\times\frac{1}{2}+1\right)}{\Delta_2\psi_2(s)d_2}\right]^2\leqslant f_2(t,x)$。所以 H_3)成立。

故定理 1 中所有条件都成立，从而边值问题(14)有唯一正解 $(x^*,y^*)\in P_h$，$h(t)=(t^{\frac{27}{14}},t^{\frac{33}{17}})$，对初值 (x_0,y_0) 做迭代序列

$$\begin{cases} x_n(t)=\displaystyle\int_0^1 G_1(t,s)\varphi_2\left[I_{0+}^{\frac{1}{3}}\left[3\left[\frac{y_{n-1}^{\frac{1}{5}}(s)}{1+t^2}+r+\frac{2Q\Gamma\left(\frac{4}{3}\right)}{405}\right]+3\left[\frac{y_{n-1}(s)}{(1+y_{n-1}(s))(1+t^{\frac{1}{2}})}+e\right]\right]\right]\mathrm{d}s+\frac{2t^{\frac{27}{14}}}{3}\omega_1, \\ y_n(t)=\displaystyle\int_0^1 G_2(t,s)\varphi_{\frac{3}{2}}\left[I_{0+}^{\frac{1}{5}}\left[3\left[\frac{x_{n-1}^{\frac{1}{3}}(s)}{3+t^4}+r+\frac{121Q^2\Gamma\left(\frac{6}{5}\right)}{2\ 484\ 300}\right]+3\left[\frac{x_{n-1}^{\frac{1}{2}}(s)}{(1+x_{n-1}^{\frac{1}{2}}(s))(4+t^2)}+e\right]\right]\right]\mathrm{d}s+\frac{10t^{\frac{33}{17}}}{13}\omega_2, \end{cases} \tag{15}$$

$n=1,2,\cdots$。当 $n\to\infty$ 时，则有 $(x_n,y_n)\to(x^*,y^*)$。

最后，取函数 $x_0(t)=t^{\frac{27}{14}}$，$y_0(t)=t^{\frac{17}{33}}$ 作为初始值，按照迭代序列式(15)进行迭代。部分数据迭代结果如图 1 所示，可以直观形象地表现出迭代序列式(15)的收敛性。

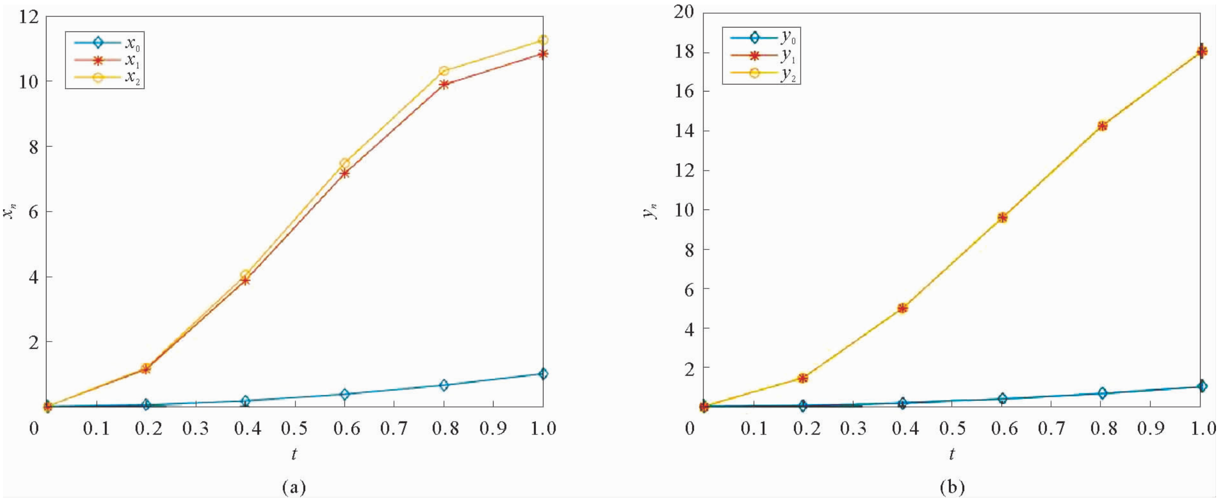

图 1 迭代序列 $x_n,y_n(n=0,1,2)$

Fig. 1 Iterative sequence $x_n,y_n(n=0,1,2)$

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